MATH2050C Selected Solutions to Assignment 4

Section 3.1

(5d) We have
$$\left|\frac{n^2-1}{2n^2+3}-\frac{1}{2}\right| = \frac{5}{2(2n^2+3)} < \frac{5}{4n^2} \ .$$
Therefore, for any number n_ε satisfying $\geq \sqrt{[5/4\varepsilon]+1}$, we have
$$\left|\frac{n^2-1}{2n^2+3}-\frac{1}{2}\right| < \varepsilon \ , \quad \forall n \geq n_\varepsilon \ .$$
So
$$\lim_{n\to\infty}\frac{n^2-1}{2n^2+3} = \frac{1}{2} \ .$$
Here $[a]$ is the integer part of a. For instance, $[1.2]=1, [5]=5, [-3.4]=-3$.

(6c) Using
$$0 < \frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \ ,$$
we see that for all $n \geq [1/\varepsilon^2]+1$,
$$\left|\frac{\sqrt{n}}{n+1}-0\right| < \frac{1}{\sqrt{n}} < \varepsilon \ , \quad \forall n \geq n_\varepsilon \ ,$$
where n_ε can be chosen to be any natural number $\geq [1/\varepsilon^2]+1$. So
$$\lim_{n\to\infty}\frac{\sqrt{n}}{n+1} = 0 \ .$$

(17) Use
$$\frac{2^n}{n!} = \frac{2}{1}\frac{2}{2}\frac{2}{3}\frac{2}{4}\cdots\frac{2}{n} < \frac{2}{1}\frac{2}{2}\frac{2}{3}\frac{2}{3}\cdots\frac{2}{3} = 2\left(\frac{2}{3}\right)^{n-2} \ , \quad \forall n \geq 4.$$
It suffices to choose n_ε such that
$$2\left(\frac{2}{3}\right)^{n-2} < \varepsilon,$$
that is,
$$n_\varepsilon > 2 + \frac{\log(\varepsilon/2)}{\log(2/3)} \ .$$
Note. In general, one can show that $\lim_{n\to\infty} a^n/n! = 0$ for every $a > 0$.

(18) It suffices to consider $\varepsilon = x/2$. Then there is some K such that $|x_n - x| < x/2, \quad \forall n \geq K$. By writing it as $-x/2 < x_n - x < x/2$, we get $x/2 < x_n < 3x/2$.

Section 3.2 (1d) We write
$$\frac{2n^2+3}{n^2+1} = \frac{2+3/n^2}{1+1/n^2}.$$

Then

$$\begin{aligned}\lim_{n\to\infty}\frac{2n^2+3}{n^2+1} &= \lim_{n\to\infty}\frac{2+3/n^2}{1+1/n^2} \\ &= \frac{\lim_{n\to\infty}(2+3/n^2)}{\lim_{n\to\infty}(1+1/n^2)} \quad \text{(by Limit Theorem)} \\ &= \frac{2}{1}=2.\end{aligned}$$

(5) Both sequences are not bounded, so they cannot be convergent.

(11) (a) Write

$$(3n^{1/2})^{1/2n} = (3^{1/2})^{1/n}n^{1/4n} = (3^{1/2})^{1/n}(4n)^{1/4n}(4^{-1/4})^{1/n} = (3^{1/2}4^{-1/4})^{1/n}(4n)^{1/4n}\ .$$

Use the known facts $\lim_{n\to\infty} a^{1/n} = 1$ $(a > 0)$ and $\lim_{n\to\infty} n^{1/n} = 1$, we have

$$\lim_{n\to\infty}(3n^{1/2})^{1/2n} = \lim_{n\to\infty} a^{1/n}(4n)^{1/4n} = \lim_{n\to\infty} a^{1/n} \lim_{n\to\infty}(4n)^{1/4n} = 1\times 1 = 1,$$

where $a = 3^{1/2}4^{-1/4}$ by Limit Theorem.

Note. Here we have used the trivial fact: $\lim_{n\to\infty} n^{1/n} = 1$ implies $\lim_{n\to\infty}(4n)^{1/4n} = 1$.

(b) (b) Let $x_n = (n+1)^{1/\log(n+1)}$. Then $\log x_n == \dfrac{1}{\log(1+n)}\log(1+n) = 1$. So this is a constant sequence $\{e, e, e, \cdots\}$ and $\lim_{n\to\infty} x_n = e$.

(12) We have

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{b^{n+1}(1+(a/b)^{n+1})}{b^n(1+(a/b)^n)} = b\frac{1+(a/b)^{n+1}}{1+(a/b)^n}\ .$$

Therefore, by Limit Theorem, and $0 < a/b < 1$,

$$\lim_{n\to\infty}\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \lim_{n\to\infty} b\frac{1+(a/b)^{n+1}}{1+(a/b)^n} = b\frac{\lim_{n\to\infty}(1+(a/b)^{n+1})}{\lim_{n\to\infty}(1+(a/b)^n)} = b\ .$$

(as $\lim_{n\to\infty}(a/b)^n = 0$ for $a/b \in (0,1)$.)

Note. We have used the fact $\lim_{n\to\infty}\alpha^n = 0$ for $\alpha \in (0,1)$. The fact was proved in class and in the text book. You may simply quote it.

Supplementary Exercise

(1). Let $p(x) = a_0 + a_1x + \cdots + a_nx^n, a_n \neq 0$, and $q(x) = b_0 + b_1x + \cdots + b_mx^m, b_m \neq 0$, be two polynomials. Consider the sequence $x_k = p(k)/q(k), k \geq 1$, (when k is large, $q(k)$ does not vanish, so you may assume that q is always non-zero). Prove that

(a) When $n = m$, $\lim_{k\to\infty} x_k = a_n/b_m$;

(b) When $n > m$, $\{x_k\}$ does not converge ; and

(c) When $n < m$, $\lim_{k\to\infty} x_k = 0$.

(a) Write

$$\frac{p(k)}{q(k)} = \frac{k^n(a_0/k^n + a_1/k^{n-1} + \cdots + a_n)}{k^m(b_0/k^m + b_1/k^{m-1} + \cdots + b_m)} = \frac{a_0/k^n + a_1/k^{n-1} + \cdots + a_n}{b_0/k^n + b_1/k^{n-1} + \cdots + b_n},$$

when $m = n$. By Limit Theorem,

$$\lim_{k\to\infty} \frac{p(k)}{q(k)} = \frac{\lim_{k\to\infty}(a_0/k^n + a_1/k^{n-1} + \cdots + a_n)}{\lim_{n\to\infty}(b_0/k^n + b_1/k^{n-1} + \cdots + b_n)} = \frac{a_n}{b_n} \ .$$

(b) WLOG let $a_n, b_m > 0$. Using the fact that $\lim_{n\to\infty}(a_0/k^n + a_1/k^{n-1} + \cdots + a_{n-1}/k) = 0$, for $\varepsilon > 0$, there is some k_0 such that

$$|a_0/k^n + a_1/k^{n-1} + \cdots + a_{n-1}/k - 0| < \varepsilon, \quad \forall n \geq n_0.$$

Choose $\varepsilon = a_n/2$, we have

$$|a_0/k^n + a_1/k^{n-1} + \cdots + a_{n-1}/k - 0| < \frac{a_n}{2}, \quad \forall k \geq k_0.$$

It follows that $a_0/k^n + a_1/k^{n-1} + \cdots + a_{n-1}/k + a_n > a_n/2$. Similarly we can find k_1 such that $b_0/k^m + b_1/k^{m-1} + \cdots b_{m-1}/k + b_m < 2b_m$ for all $k \geq k_1$. Thus,

$$\frac{p(k)}{q(k)} = \frac{k^n(a_0/k^n + a_1/k^{n-1} + \cdots + a_n)}{k^m(b_0/k^m + b_1/k^{m-1} + \cdots + b_m)} > \frac{k^n a_n/2}{k^m 2b_m} = \frac{a_n}{4b_m} k^{n-m}$$

for all $k \geq \max\{k_0, k_1\}$. Now, given $M > 0$, it is clear there is some K such that

$$\frac{p(k)}{q(k)} \geq \frac{a_n}{4b_m} k^{n-m} > M$$

for all $k \geq K$. Indeed, it suffices to choose to be any natural number satisfying

$$K \geq k_0, k_1, \ \left(\frac{4b_m M}{a_n}\right)^{1/(n-m)} \ .$$

We conclude that $\{x_k\}$ is not convergent, in fact,

$$\lim_{k\to\infty} x_k = \infty.$$

(When $a_n b_m < 0$, it is $-\infty$ instead of ∞.)

(c) Leave it to you.